A MODERN PROOF OF CHEVALLEY'S THEOREM ON ALGEBRAIC GROUPS

BRIAN CONRAD

1. Introduction

Let k be a field, and let G be an *algebraic group* over k, by which we mean a connected smooth k-group scheme (*not* necessarily affine). Recall that such a G is automatically separated, finite type, and geometrically integral over k [4, Exp $\mathrm{VI_A}$, 0.3, 2.1.2, 2.4]. The most important classes of algebraic groups are the affine algebraic groups (also called *linear algebraic groups*, since affine algebraic groups coincide with the closed algebraic subgroups of the matrix groups $\mathrm{GL}(n)_{/k}$ [15, Thm 3.4]) and the proper algebraic groups (also called *abelian varieties*). Other types of algebraic groups do arise naturally.

For example, if X is a proper (possibly singular) scheme over a perfect field k then the reduced connected component $(\mathrm{Pic}^0_{X/k})_{\mathrm{red}}$ of its relative Picard scheme is a typically non-proper non-affine algebraic group over k; when X is an algebraic curve then such algebraic groups are the so-called generalized Jacobians of geometric class field theory [13]. In another direction, if R is a discrete valuation ring with fraction field K and residue field k and if A is an abelian variety over K with Néron model $\mathcal{A}$ over R, the connected component of the closed fiber of $\mathcal{A}$ is an algebraic group over k which usually is neither affine nor proper (properness of the connected component of the closed fiber is *equivalent* to good reduction, by EGA IV_3, 15.6.7,15.6.8 and [1, 1.2/8]). Thus, it is important to understand something about the general structure of general algebraic groups.

Chevalley's Theorem asserts that every algebraic group over a perfect field is 'built up' from a linear algebraic group and an abelian variety (in a way we will make precise shortly). This is an extremely important result. For example, the proof of the Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties (see [14, Thm 1] and [1, 7.4/5]) relies heavily on this general structure theorem. Geometric class field theory (as in [13]), which classifies rational maps from an algebraic curve to commutative algebraic groups, also depends on Chevalley's Theorem. It came as somewhat of a surprise to the author to find that in the published literature there does not exist a proof of Chevalley's Theorem in modern language, even though the theorem is widely used. The purpose of this note is to present a proof based on scheme theory rather than Weil's Foundations [16].

The published proofs of Chevalley's Theorem, by both Chevalley [2] and Rosenlicht [12], are written with very archaic terminology which is no longer used. For example, [12] seems to be unreadable for those not familiar with Weil's Foundations. However, the underlying method in [2] is almost penetrable, so below we present a modern translation of this proof. There are two sources of technical difficulties with Chevalley's proof. First, he employs an old-style notion of "algebraic families of divisors" which sits somewhere between the notions of Weil divisor and relative effective Cartier divisor. It requires a little care to translate this into modern terminology while bypassing Chevalley's papers on the old theory of divisor families. Basic results on invertible sheaves (such as in [9]) will supply the tools we need.

Date: April 2, 2002.

1991 *Mathematics Subject Classification.* Primary 20G15; Secondary 14A10.

Key words and phrases. Chevalley's theorem, algebraic groups, descent theory.

This work was supported by the NSF. Since this manuscript was originally written without intention of publication, I would like to thank Johan de Jong for helping to make it more widely available and Dinakar Ramakrishnan for suggesting that it be published so as to be permanently available. I am also grateful to Angelo Vistoli and Ravi Vakil for asking questions which led to locating an error in an earlier version.

The second source of difficulties is Chevalley's extensive use of birational isomorphisms in place of actual morphisms. It is a lot easier to work with Weil divisors and invertible sheaves when we use morphisms. We can circumvent the intensive use of birational isomorphisms because we are able to invoke the Nagata Compactification Theorem (in the case where the base is an algebraically closed field). This theorem of Nagata's [10], [11] asserts that any separated finite type scheme over a noetherian base admits an open immersion into a proper scheme over the base. Nagata's arguments are very difficult to follow, but Deligne has worked out in terms of schemes what Nagata was saying [3] and Lütkebohmert has published a modern proof of Nagata's theorem [6], so there is no harm in using Nagata's theorem. This result, coupled with the use of faithfully flat descent (a technique not available to pre-Grothendieck mathematicians), is probably the main reason we can give a proof which is much shorter than the classical proofs.

If one is only interested in proving Chevalley's Theorem in the quasi-projective case (without knowing in advance that this condition is automatically satisfied, as follows from Chevalley's Theorem; cf. Corollary 1.2), then Nagata's Theorem is not needed in the proof of Chevalley's Theorem. This case includes closed fibers of Néron models of abelian varieties, since such Néron models are quasi-projective over the Dedekind base by construction. Thus, for the purpose of proving the Néron-Ogg-Shafarevich criterion, the Nagata Compactification Theorem is not needed.

Let's now give the precise statement of Chevalley's Theorem. First, some preparations will be needed.

If G is an algebraic group over a field k and H is a closed algebraic subgroup (i.e., a closed subgroup scheme in G which is smooth and connected), we say that H is *normal* in G if $H(T)$ is a normal subgroup of $G(T)$ for every k-scheme T. This property can be checked after base change to the algebraic closure of k, and over an algebraically closed field k this is equivalent to $H(k)$ being a normal subgroup of $G(k)$ (one checks this by reducing to a 'universal' case involving only reduced k-schemes of finite type). Using [4, Exp $\mathrm{VI_A}$, Thm 3.2], we see that for a closed normal algebraic subgroup H in an algebraic group G, there exists a finite type k-scheme G/H which represents the fppf quotient sheaf of G by H on the category of k-schemes (so G/H is a k-group scheme) and the canonical map of group schemes $G \to G/H$ is faithfully flat with scheme-theoretic kernel H. Clearly G/H is connected. By combining fppf descent and the isomorphism $G \times_{G/H} G \simeq H \times_k G$ (defined by $(g_1, g_2) \mapsto (g_1^{-1}g_2, g_2)$), we see that G/H is smooth and separated over k since H is, so G/H is an algebraic group in the sense we have defined. When G is a linear algebraic group, it follows by fpqc descent from an algebraic closure of k and [5, Thm 11.5, §11] that G/H is again a linear algebraic group.

The theorem linking up arbitrary algebraic groups with linear algebraic groups and abelian varieties is:

Theorem 1.1. (Chevalley) *Let k be a perfect field and G an algebraic group over k. Then there exists a unique normal linear algebraic closed subgroup H in G for which G/H is an abelian variety. That is, there is a unique short exact sequence of algebraic groups*

$$1 \to H \to G \to A \to 1$$

with H linear algebraic and A an abelian variety. The formation of H commutes with base change to an arbitrary perfect field extension over k.

We should mention that due to the uniqueness in Chevalley's Theorem, Galois descent immediately reduces the problem to the case of an algebraically closed ground field. Thus, in our proof of Chevalley's Theorem we will only consider the case of algebraically closed k. As for the case of possibly non-perfect k, one can get a statement involving possibly non-smooth affine closed subgroup schemes. See [1, 9.2/1] for more details. This asserts in particular that an algebraic group over a field always contains a (possibly non-smooth) connected affine closed normal subgroup scheme such that the quotient is an abelian variety.

One nice consequence of Chevalley's Theorem is the following well-known fact:

Corollary 1.2. *Any algebraic group G over a field k is necessarily quasi-projective.*

Proof. By EGA IV_3, 9.1.5, we may assume k is algebraically closed. By EGA II, 5.3.6, we may assume G is connected, hence an algebraic group in the sense we have defined. Chevalley's Theorem yields a faithfully flat algebraic group map $G \to A$ with A an abelian variety and with affine kernel H. By Lemma 2.1 below, it follows that $G \to A$ is an affine map of finite type. Since abelian varieties are projective [9, p.62], we conclude from EGA II, 5.3.4(i), 5.3.3, 5.3.4(ii) that $G \to \mathrm{Spec}(k)$ is a quasi-projective map in the usual sense. ■

Notation

We usually denote projection maps by p_i, for projection onto the ith factor of a product. We almost never use rational maps, so when they do arise we denote them by a solid arrow rather than a dotted arrow (with a warning that the map is just a rational map).

If G is an algebraic group over k, we let $e \in G(k)$ denote the identity element, and $m : G \times_k G \to G$ denote the multiplication map. If G is merely a separated finite type k-group scheme, we denote by G^0 the connected component of e (an algebraic group if G is smooth over k).

If $X \to S$ is a map of schemes, this is sometimes denoted $X_{/S}$. Also, X_{red} denotes the underlying reduced subscheme of a scheme X. If X and Y are S-schemes with S a field which is understood from context, we sometimes write $X \times Y$ in place of $X \times_S Y$ when no confusion will result. If $\mathcal{F}$ is an $\mathcal{O}_{X\times_S Y}$-module and $y \in Y$ is a point, we write $\mathcal{F}_y$ for the pullback to the fiber $X \times_S \mathrm{Spec}(k(y))$. If $S = \mathrm{Spec}(k)$ and $k = k(y)$, then we identify $\mathcal{F}_y$ with a sheaf on X in the evident manner. When X is a *variety* over a field k (i.e. a separated finite type k-scheme which is geometrically integral), we denote by $k(X)$ the function field of X.

If V is a finite-dimensional vector space over a field k, we denote by $\mathbf{P}(V)$ the projective space $\mathrm{Proj}(S(V))$, where $S(V)$ is the symmetric algebra of V over k. Recall that the universal mapping property of this for k-schemes T is that a map $T \to \mathbf{P}(V)$ over k is equivalent to the choice of an isomorphism class of invertible quotient sheaf of $\mathcal{O}_T \otimes_k V$ on T. In particular, $\mathbf{P}(V)(k)$ corresponds to the non-zero linear functionals on V (not the lines in V), modulo $k^\times$-scaling. Also, for a given invertible quotient sheaf $\mathcal{O}_T \otimes_k V \twoheadrightarrow \mathcal{L}$ and $t \in T(k)$, the map $T \to \mathbf{P}(V)$ sends t to the functional $V \simeq k(t) \otimes_k V \twoheadrightarrow \mathcal{L}/\mathfrak{m}_t \simeq k$ (where this last isomorphism is ambiguous up to $k^\times$-scaling, which doesn't matter). We write $V^\vee$ and $\mathcal{L}^\vee$ for the duals of a vector space V and an invertible sheaf $\mathcal{L}$, so $\mathbf{P}(V^\vee)$ classifies "lines in V".

If $U \hookrightarrow Z$ is an open immersion of smooth varieties over an algebraically closed field k and D is a Weil divisor on U, we define the *extension by zero* of D to be the Weil divisor on Z whose components are the ones with the same generic points as those of D and with same multiplicities. If $Z' \hookrightarrow Z$ is a closed immersion of schemes, we denote by $\mathcal{I}(Z')$ the associated quasi-coherent ideal sheaf on Z.

2. Preliminaries

For the convenience of the reader, we begin by recalling (with proof) some basic facts about algebraic groups which we shall need to use. We begin with a basic descent theory lemma which (as we remarked in the introduction) is one of the main reasons we can give a short proof of Chevalley's theorem.

Lemma 2.1. *Let $f : G \to G'$ be a surjective morphism of algebraic groups over a field k, with scheme-theoretic kernel H. Let $\mathbf{P}$ be a property of scheme morphisms which is local for the fppf topology on the target (e.g., finite, affine, smooth, flat). Then f has property $\mathbf{P}$ if and only if $H \to \mathrm{Spec}(k)$ has property $\mathbf{P}$.*

Proof. As we noted in the introduction, both G and G' are automatically of finite type over k, so f is finite type. Moreover, since the source and target of f are smooth varieties and all fibers have the same dimension, by [7, Cor to Thm 23.1] we know that f must be flat and hence faithfully flat. Thus, f is an fppf morphism. But if we make a base change on f by $G \to G'$ and use the isomorphism

$$G \times_{G'} G \simeq H \times_k G$$

over G (via p_2) defined by $(g_1, g_2) \mapsto (g_1^{-1}g_2, g_2)$, we see that f satisfies $\mathbf{P}$ if and only if $p_2 : H \times_k G \to G$ satisfies $\mathbf{P}$, and this in turn is equivalent to $H \to \mathrm{Spec}(k)$ satisfying property $\mathbf{P}$. ■

Lemma 2.2. *Let G be an algebraic group over a field k, A an abelian variety over k. Then any map of k-schemes $f : G \to A$ sending the identity $e \in G(k)$ to 0 is a map of algebraic groups.*

Proof. We may base change to the case where k is algebraically closed. We now explain how to reduce ourselves to the case of the classical rigidity lemma for complete varieties. Consider the map $\varphi : G \times_k G \to A$

given by $(x, y) \mapsto f(x, y) - f(x) - f(y)$. Note that $\varphi(G \times \{e\}) = \varphi(\{e\} \times G) = \{0\}$. If we can show that $\varphi = \psi \circ p_2$ for some k-morphism $\psi : G \to A$, then $\varphi(x, y) = \psi(y) = \varphi(e, y) = 0$ and we'll be done.

We now consider more generally two finite type separated k-varieties V and W with W smooth, and a k-morphism $\varphi : V \times_k W \to A$ with $\varphi(V \times \{w_0\}) = \{a_0\}$ for some $w_0 \in W(k)$, $a_0 \in A(k)$. We will show that $\varphi = \psi \circ p_2$ for some $\psi : W \to A$ over k. Choose $v_0 \in V(k)$ and define $\psi : W \simeq \{v_0\} \times W \hookrightarrow V \times_k W \xrightarrow{\varphi} A$. It suffices to show $\varphi(v, w) = \varphi(v_0, w)$ for all $v \in V(k)$, $w \in W(k)$, so then $\psi \circ p_2$ and φ coincide on k-rational points and therefore are equal.

Choose $v \in V(k)$. Since $\dim V > 0$ without loss of generality, by [9, Lemma, p. 56] there exists an irreducible closed curve C in V through v_0 and v, so base changing to the normalization of C allows us to assume that V is a *smooth* connected separated curve (this smoothness is where we really need k algebraically closed). Consider the (unique) open immersion $V \hookrightarrow \overline{V}$ with $\overline{V}$ a projective smooth connected curve over k. We have a rational map $\overline{\varphi} : \overline{V} \times_k W \to A$. Since $\overline{V} \times_k W$ is smooth, hence regular in codimension 1, the domain of φ contains all codimension 1 points, by applying the valuative criteria for properness to $A \to \mathrm{Spec}(k)$. Making full use of the smoothness of $\overline{V} \times_k W$ and the fact that A is an abelian variety, Weil's Extension Theorem [1, 4.4/1] implies that $\overline{\varphi}$ is a k-morphism (not just a rational map). Since $\varphi(V \times \{w_0\}) = \{a_0\}$, clearly $\overline{\varphi}(\overline{V} \times \{w_0\}) = \{a_0\}$. Thus, we're reduced to the case where $V \to \mathrm{Spec}(k)$ is *proper*. Now we can apply the classical rigidity lemma [9, p. 43]. ∎

Lemma 2.3. *Let $f : G \to A$ be a map of k-varieties from a linear algebraic group G over k to an abelian variety A over k. Then f is constant (i.e., f factors through $f(e) \in A(k)$).*

Proof. Performing a suitable translation, we can assume $f(e) = 0$. We now claim that $f = 0$. For this, we may assume k is algebraically closed. By Lemma 2.2, f is an algebraic group map. Since k is algebraically closed, $(\ker(f))^0_{\mathrm{red}}$ is a closed normal algebraic subgroup of G. Replacing G by the quotient by this subgroup, we can assume that $\ker(f)$ is finite (topologically just a single point). By a constructibility argument [4, VI_B, 1.2], f has closed image, so we can replace A with the (smooth) scheme-theoretic image $f(G)$ so as to assume that f is surjective. Since $f : G \to A$ has finite kernel, by Lemma 2.1 it follows that f is finite, so $G \to \mathrm{Spec}(k)$ is proper. Since this map is also affine, so $G = \mathrm{Spec}(k)$. ∎

Lemma 2.3 shows that the linear algebraic group H in Chevalley's Theorem is unique; moreover, all algebraic group maps $H' \to G$, with H' linear algebraic, factor uniquely through H. As for existence, we begin by showing that it suffices to prove that Lemma 2.3 characterizes the *linear* algebraic groups. Chevalley's proof of this point uses Albanese varieties. This is unnecessary, thanks to fppf descent:

Lemma 2.4. *Let k be an algebraically closed field. Suppose it is proven that an algebraic group over k having no non-constant maps to an abelian variety is necessarily a linear algebraic group. Then Chevalley's Theorem is true over k.*

Proof. We induct on $\dim G$, the case of dimension 0 being clear. Our hypothesis also allows us to assume that there is a non-constant map $\varphi : G \to A$ with A an abelian variety over k. Performing a translation, using Lemma 2.2, and shrinking A as in the proof of Lemma 2.3, we can assume that φ is a faithfully flat algebraic group map. Let $H = (\ker(\varphi))^0_{\mathrm{red}}$. Since $G/H \to A$ is faithfully flat with kernel finite over k, it follows from Lemma 2.1 that $G/H \to A$ is finite, so G/H is proper over k and therefore an abelian variety. Since φ is non-constant, $H \neq G$, so $\dim H < \dim G$. By induction, there exists a unique closed normal linear algebraic subgroup Γ in H with H/Γ an abelian variety. Since Γ is stable under $\mathrm{Aut}(H)$ by the uniqueness of Γ, we see by considering k-rational points that Γ is normal in G. Since $G/H \simeq (G/\Gamma)/(H/\Gamma)$ (think of fppf sheaves), so $G/\Gamma \to G/H$ is fppf with kernel H/Γ proper over k, it follows from Lemma 2.1 that $G/\Gamma \to G/H$ is proper, so $G/\Gamma \to \mathrm{Spec}(k)$ is proper and thus is an abelian variety. ∎

Thanks to Lemma 2.4, the proof of Chevalley's Theorem is reduced to the following: fix an algebraically closed field k and an algebraic group G over k which admits no non-constant maps to an abelian variety over k. Then we need to prove that G is linear algebraic. Roughly speaking, we will construct a finite-dimensional k-vector space W out of rational functions on G and an algebraic group map $\rho : G \to \mathrm{GL}(W)$ with $\ker(\rho) = \{e\}$ on the level of k-rational points. Then $\ker(\rho)$ is finite over k and $\rho(G) \subseteq \mathrm{GL}(W)$ is a closed algebraic subgroup (and hence affine). The map $G \to \rho(G)$ is faithfully flat, hence finite (by considering the kernel and using Lemma 2.1), so G is affine over k as desired.

How will we construct such a representation ρ? For each effective Weil divisor D on G, we will construct an algebraic representation $\rho_D : G \to \mathrm{GL}(W_D)$, with W_D related to the space of rational functions on G with poles 'no worse' than D, and we will check that the set-theoretic intersection of the $\ker(\rho_D)$'s over all D is $\{e\}$. By the noetherian property of G, it follows that this intersection is $\{e\}$ for finitely many $\rho_{D_1}, \dots, \rho_{D_r}$. Defining ρ to be the direct sum of the ρ_{D_i} will then finish the proof of Chevalley's Theorem.

Unfortunately, for an effective Weil divisor D on G, the sheaf $\mathcal{L}(D)$ generally has an infinite-dimensional k-vector space of global sections. In order to get around this, we want to 'approximate' G by a k-variety U which is 'almost complete' so that the space of global sections of each invertible sheaf on U is finite-dimensional (this will roughly be what the W_D's are). However, our 'approximating' k-variety should also be smooth, so invertible sheaves relate well to Weil divisors. Since we do not want to have to invoke results related to resolution of singularities in charactertistic p, we can't expect to work with complete k-varieties. We begin the next section by defining the 'almost complete' smooth k-variety which we will use.

From now on, it is to be understood that all varieties and morphisms are over a fixed algebraically closed k. Also, G is now fixed as an algebraic group admitting no non-constant map to an abelian variety.

3. Passing to a Semi-complete Variety

Since G is separated and finite type over k, by the Nagata Compactification Theorem we can realize G as an open subscheme of a proper k-scheme X. Replacing X by the scheme-theoretic closure of G and then normalizing, we can assume that X is a proper *normal* k-variety. Let U denote the *open* regular locus of X [7, Cor to Thm 30.5]. Then $G \subseteq U$ with U smooth, and due to the normality of X we have that the complement $X - U$ is of codimension at least 2 in X. Chevalley calls such a U 'semi-complete'. That is, although the smooth variety U isn't likely to be complete, U is 'close enough' to X that it acts as if it were complete.

For example, pick an invertible sheaf $\mathcal{L}$ on U, so $\mathcal{L} \simeq \mathcal{L}(D)$ for a Weil divisor D on U (this D exists since U is a smooth variety). Using the complementary codimension 2 property and the invertibility of $\mathcal{L}$, the pushfoward $i_*(\mathcal{L})$ by $i : U \hookrightarrow X$ is a coherent sheaf on X (see EGA IV$_2$, 5.11.4(*ii*)). In particular, $\Gamma(U, \mathcal{L}) \simeq \Gamma(X, i_*(\mathcal{L}))$ is a *finite-dimensional* k-vector space. Clearly the identification of function fields $k(X) \simeq k(U)$ is compatible with formation of divisors of (non-zero) rational functions. We conclude that for any effective Weil divisor D on U, the k-vector space of $f \in k(U)$ with $f = 0$ or $f \neq 0$ and $\mathrm{div}(f) \geq -D$ is a finite-dimensional k-vector space and when $f, g \in k(U)^\times$ have the same divisor, then $f = cg$ for some $c \in k^\times$. These are the basic properties of U which make it behave almost as if it were complete, and this is why U will be more convenient to use than G for the study of $k(G) \simeq k(U)$.

Note that U has no non-constant maps to an abelian variety, since the dense open $G \subseteq U$ has this property. We will be studying certain families of Weil divisors on U, so the following Lemma is where the peculiar mapping property of U with respect to abelian varieties will be employed.

Lemma 3.1. *Let Y be a smooth variety which has no non-constant map to an abelian variety. Let T be any variety, $\mathcal{L}$ an invertible sheaf on $T \times_k Y$. The invertible sheaves $\mathcal{L}_t$ on Y (for $t \in T(k)$) are all isomorphic.*

Proof. Fix $t_0 \in T(k)$. By [9, Prop, p. 56], there exists an irreducible closed curve C on T through t_0 and a chosen $t \in T(k)$. Base changing to the normalization of C, we can assume that T is a smooth separated connected curve. Let $\overline{T}$ be the unique smooth projective connected curve with T as an open subscheme. Since $T \times Y$ is smooth, we can identify $\mathcal{L}$ with $\mathcal{L}(D)$ for some Weil divisor D on $T \times Y$. Extending this by zero to a Weil divisor on the smooth variety $\overline{T} \times Y$ and passing to the associated invertible sheaf, we are reduced to the case in which T is a projective smooth connected curve.

Choose $y_0 \in Y(k)$, let $\mathcal{L}' = \mathcal{L} \otimes p_2^*(\mathcal{L}_{t_0})^{-1} \otimes p_1^*(\mathcal{L}_{y_0})^{-1}$, and choose trivializations of $\mathcal{L}'_{t_0}$ on Y and $\mathcal{L}'_{y_0}$ on T. We will show that all $\mathcal{L}'_y$ on T are trivial (for $y \in Y(k)$). Since Y is a variety and T is a complete variety, it then will follow from the See-saw theorem [9, Cor 6, p. 54] that $\mathcal{L}' \simeq p_1^*(\mathcal{N})$ for some invertible sheaf $\mathcal{N}$ on Y. Then $\mathcal{L}'_t \simeq \mathcal{N}$ for all $t \in T(k)$, so taking $t = t_0$ shows that $\mathcal{N}$ is trivial. Since $\mathcal{L}'_t \simeq \mathcal{L}_t \otimes \mathcal{L}_{t_0}^{-1}$ for $t \in T(k)$, we'd be done.

We now need to get a handle on the fibers of $\mathcal{L}'$ over $Y(k)$. Recall that we chose a trivialization of $\mathcal{L}'_{t_0}$. Since T is a projective smooth connected curve over an algebraically closed field k, Y is a variety, and $\mathcal{L}'_{y_0}$ is trivialized, the theory of the Jacobian [8, Thm 1.2] yields a unique map $\varphi : Y \to \mathrm{Pic}^0_{T/k}$ sending y_0 to 0 and pulling the universal invertible sheaf on $T \times \mathrm{Pic}^0_{T/k}$ (trivialized over t_0) back to to $\mathcal{L}'$ on $T \times Y$ (trivialized over t_0). But the Jacobian $\mathrm{Pic}^0_{T/k}$ is an abelian variety, so φ is constant, hence identically 0. This says exactly that $\mathcal{L}'_y$ is trivial for all $y \in Y(k)$, as desired. ■

This Lemma implies that if we construct an invertible sheaf on $G \times U$ with trivial fiber on $p_1^{-1}(e)$, then the fibers on $p_1^{-1}(g) \simeq U$ are trivial for all $g \in G(k)$. This will produce many rational functions for us if we have a way to standarize the associated principal Weil divisors along these fibers (recall that on U, a rational function is determined mod $k^\times$ by its Weil divisor). The invertible sheaves of interest on $G \times U$ will arise by extending invertible sheaves on $G \times G$, as we shall now see.

Choose any effective Weil divisor D on G (viewed as a closed subscheme of G), and set $\mathcal{L} = \mathcal{L}(D)$ to be the inverse of its ideal sheaf. Define $\mathcal{L}_D^G = m^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})^{-1}$ on $G \times G$. More precisely, $\mathcal{L}_D^G = \mathcal{L}(m^{-1}(D) - H \times D)$ (note that since $D \hookrightarrow G$ is cut out by an invertible ideal sheaf and $m : G \times G \to G$ is faithfully flat, $m^{-1}(D) \hookrightarrow G \times G$ is cut out by the pullback *invertible* ideal sheaf, so this is an effective Weil divisor on $G \times G$). We claim that for $g \in G(k)$, the pullback of $\mathcal{L}_D^G$ to $\{g\} \times G \simeq G$ is isomorphic to $\ell_g^*(\mathcal{L}) \otimes \mathcal{L}^{-1} \simeq \mathcal{L}(\ell_g^{-1}(D) - D)$, where $\ell_g : G \simeq G$ is the 'left multiplication by g' map. More precisely, we claim that when $G \times G$ is viewed as a G-scheme via p_1, the ideal sheaf of the closed subscheme $m^{-1}(D) \hookrightarrow G \times G$ is of formation compatible with base change on G. Since G is integral, by [1, 8.2/6(*iii*)] it is enough to show that $m^{-1}(D) \hookrightarrow G \times G$ is an effective Cartier divisor relative to G, or equivalently that $m^{-1}(D)$ doesn't set-theoretically contain any $\{g\} \times G$ for $g \in G(k)$. This is clear. Note in particular that $\mathcal{L}_D^G$ has trivial fiber over e, so by Lemma 3.1, $\mathcal{L}_D^G$ has trivial fiber over all $g \in G(k)$.

We want to extend $\mathcal{L}_D^G$ to $G \times U$ in a useful manner. We begin with a naive definition, and we will have to exercise some care to make sure that this definition behaves well with respect to pullback to p_1-fibers over $G(k)$. Let D_1 on U denote the extension by zero Weil divisor extending D on G. Similarly, let D_2 on $G \times U$ denote the extension by zero Weil divisor extending $m^{-1}(D)$ on $G \times G$. We define $\mathcal{L}_D^U = \mathcal{L}(D_2 - G \times D_1)$. The fibers of this sheaf over $G(k)$ will be our source of rational functions on U. In order to understand these fiber sheaves, we need to pay careful attention to the Weil divisors along the fibers. Note that D_2 does not set-theoretically contain any $\{g\} \times U$ for $g \in G(k)$, so the scheme-theoretic pullback $(D_2)_g$ to a closed subscheme of U makes good sense as a Weil divisor and $(\mathcal{L}_D^U)_g \simeq \mathcal{L}((D_2)_g - D_1)$ on U (again using [1, 8.2/6(*iii*)]). In concrete terms, what is this? More to the point, what is $(D_2)_g$? Certainly $(D_2)_g$ on U contains the extension by zero of $(m^{-1}(D))_g = \ell_g^{-1}(D)$. We claim that this extension by zero coincides with $(D_2)_g$. Roughly speaking, we are claiming that the naive 'extension by zero' happens to commute with passage to $G(k)$-fibers in our particular setting.

In more geometric terms, our assertion is:

Lemma 3.2. *The dense open immersion of subschemes $m^{-1}(D) \hookrightarrow D_2$ in $G \times U$ remains a dense open immersion on each fiber $\{g\} \times U$ for $g \in G(k)$.*

Proof. Since effective relative Cartier divisors behave well with respect to base change and the property of an open immersion being dense is preserved after flat base change, we see that $m^{-1}(D) \hookrightarrow D_2$ remains dense when we pass to the *generic* fiber over the variety G. Thus, by EGA IV_3, 9.5.3, there exists a non-empty open $W \subseteq G$ so that for all $g \in W(k)$, $\ell_g^{-1}(D) = (m^{-1}(D))_g \hookrightarrow (D_2)_g$ is a dense open.

Assuming that the open immersion $\ell_g^{-1}(D) \hookrightarrow (D_2)_g$ is dense for *some* $g \in G(k)$ (a hypothesis we have just seen to be satisfied for many points in $G(k)$), let's show that the same then holds for *all* points $g'g \in G(k)$ for $g' \in G(k)$ arbitrary, from which it follows that this open immersion is dense on all fibers over $G(k)$, as desired. We just have to be a little careful in order to carry over some translation arguments on G over to U (where the translation maps are just birational maps).

The map $\ell_g : G \simeq G \hookrightarrow X$ is a dense open immersion and is a birational isomorphism from X to X. By the valuative criteria for properness and the normality of X (hence regularity in codimension 1), this map extends to a k-morphism $L_g : U_g \to X$ for some open $U_g \subseteq X$ containing G, with U_g containing all codimension 1 points of X. Note that there are only finitely many of these outside of G. Let $x \in U_g$ be a codimension 1 point, so $\mathcal{O}_{U_g,x} \subseteq k(U_g) \overset{L_g^*}{\simeq} k(X)$ is identified with a valuation ring in $k(X)$. Due to the properness of $X_{/k}$, this dominates a unique local ring $\mathcal{O}_{X,L_g(x)}$, which we claim is *one-dimensional.* That is, we claim that $L_g(x)$ is a codimension 1 point of X. For clarity, let X_1 and X_2 respectively denote the source and target X's for the map L_g. Since L_g extends the automorphism ℓ_g of G and each valuation ring of $k(U_g)$ has at most one center on U_g, certainly $L_g(x)$ can't be a codimension 1 point in G. Combining this with the fact that there are only finitely many codimension 1 points $\{x_1, \dots, x_n\}$ in X not in G (and all of which lie in U_g), what we are really claiming is that L_g induces a bijection on codimension 1 points, necessarily preserving those inside of G.

We argue in a backwards manner. Choose a codimension 1 point x_i in X_2 not in G. Since $L_g^* : k(X_2) \simeq k(U_g)$ is an isomorphism and $U_g \subseteq X_1$ with X_1 proper over k, there is a unique $y_i \in X_1$ for which $\mathcal{O}_{X_1,y_i}$ is dominated by the (discrete) valuation ring $\mathcal{O}_{X_2,x_i}$ of $k(X_1) \simeq k(U_g) \simeq k(X_2)$. Thus, L_g extends to be defined in a neighborhood of y_i, taking y_i to x_i. Certainly y_i is not the generic point of X_1. But since $\overline{\{y_i\}}$ dominates the codimension 1 subvariety $\overline{\{x_i\}}$, y_i must be a codimension 1 point in X_1 (so $y_i \in U_g$). The map ℓ_g takes G isomorphically to itself, so since $x_i \notin G$, necessarily $y_i \notin G$. In other words, for each codimension 1 point $x_i \in X_2$ not in G, there exists another such point $y_i \in X_1$ not in G with $L_g(y_i) = x_i$. There are only finitely many codimension 1 points in X not in G, all lying in U_g and U, so since L_g extends an automorphism ℓ_g of G, it follows that $L_g : U_g \to U$ does indeed induce a bijection on codimension 1 points outside of G.

Since valuation rings in a field are maximal with respect to domination, L_g takes each codimension 1 point of U_g to a codimension 1 point of X and induces an *isomorphism* on the level of their (discrete valuation) local rings, so L_g is an open immersion on an open neighborhood around each codimension 1 point. If we shrink U_g a little bit around G and the finitely many codimension 1 points of X outside of G, then L_g is quasi-finite and has image inside of U. But U_g and U are separated integral varieties over k and U is normal, so by Zariski's Main Theorem [EGA IV$_3$, 8.12.10] it follows that $L_g : U_g \to U$ is an *open immersion.*

We now use L_g to show that for any $g' \in G(k)$, the open immersion $\ell_{g'g}^{-1}(D) \hookrightarrow (D_2)_{g'g}$ is dense. It is enough to check this around each codimension 1 point of U, hence on $U_g \subseteq U$. Since

$$(D_2)_{g'g} = ((r_g \times 1)^{-1}(D_2))_{g'} = ((r_g \times 1)^{-1}(\overline{m^{-1}(D)}))_{g'},$$

where $r_g \times 1 : G \times U \simeq G \times U$ is $(x, u) \mapsto (xg, u)$ and $\overline{\cdot}$ denotes scheme-theoretic closure inside of $G \times U$, it follows that

$$(D_2)_{g'g} = (\overline{(r_g \times 1)^{-1}(m^{-1}(D))})_{g'} = (\overline{(1 \times \ell_g)^{-1}(m^{-1}(D))})_{g'}$$

on U. In order to compare this with the closure of $\ell_{g'g}^{-1}(D)$ on U, it is enough to compare on U_g. But $1 \times L_g : G \times U_g \hookrightarrow G \times U$ is an open immersion extending

$$1 \times \ell_g : G \times G \simeq G \times G \hookrightarrow G \times U$$

and $m^{-1}(D) \subseteq G \times U$ has all of its generic points inside of $G \times G$. Combining this with the fact that L_g induces a bijection on codimension 1 points, preserving those which are inside of G, we compute

$$(D_2)_{g'g}|U_g = (\overline{(1 \times \ell_g)^{-1}(m^{-1}(D))}|G \times U_g)_{g'} = (\overline{(1 \times L_g)^{-1}(m^{-1}(D))}|G \times U_g)_{g'} = ((1 \times L_g)^{-1}(\overline{m^{-1}(D)}))_{g'},$$

which is equal to

$$L_g^{-1}((\overline{m^{-1}(D)})_{g'}) = L_g^{-1}(\overline{\ell_g^{-1}(D)}) = \overline{L_g^{-1}(\ell_{g'}^{-1}(D))}|U_g = \overline{\ell_g^{-1}(\ell_{g'}^{-1}(D))}|U_g = \overline{\ell_{g'g}^{-1}(D)}|U_g,$$

as desired (here, all but the first $\overline{\cdot}$ denote scheme-theoretic closure inside of U). ■

To summarize our findings so far, if we define the Weil divisors D_1 on U and D_2 on $G \times U$ to be the extensions by zero of D on G and $m^{-1}(D)$ on $G \times G$ respectively, then for $g \in G(k)$, the Weil divisor $(D_2)_g - D_1$ on U is the extension by zero of $\ell_g^{-1}(D) - D$ on G and also has associated sheaf $\mathcal{L}((D_2)_g - D_1) \simeq$

$\mathcal{L}(D_2 - G \times D_1)_g$. But Lemma 3.1 implies that the isomorphism class of this invertible sheaf on U is *independent* of $g \in G(k)$, so since $(D_2)_e - D_1$ is the extension by zero of $\ell_e^{-1}(D) - D = 0$, we conclude that for *all* $g \in G(k)$, $(D_2)_g - D_1$ is a principal Weil divisor on U.

Since U is 'semi-complete', we obtain functions $f_g \in k(U)^\times$, uniquely determined modulo $k^\times$, satisfying the condition $\mathrm{div}(f_g) = (D_2)_g - D_1$. Note that these functions all lie in the finite-dimensional k-vector space $\Gamma(U, \mathcal{L}(D_1))$. Thus, the span of these f_g's is a *finite-dimensional* k-vector space $V_D \subseteq k(U) = k(G)$. Since the f_g's are well-defined up to $k^\times$-scaling, we see that V_D is well-defined in terms of the effective Weil divisor D. It is these spaces V_D (for variable effective Weil divisors D on G) which will give rise to the desired linear algebraic representations of G.

4. Analysis of Rational Functions

We want to make $G(k)$ act on V_D in an 'algebraic' manner. Of course, the elements $f_g \in V_D$ are not quite well-defined, but the lines they generate are well-defined. For our purposes, pick choices of f_g for each $g \in G(k)$ (in the end, these choices won't matter). We do have intrinsic points $[f_g]$ in $\mathbf{P}(V_D^\vee)(k)$. Letting $G(k)$ act on V_D through a sort of 'left-regular representation', we will check that V_D is stable and that the resulting map $G(k) \to \mathrm{Aut}(\mathbf{P}(V_D^\vee))(k) = \mathrm{PGL}(V_D^\vee)(k)$ is induced by a map of algebraic groups

$$\rho_D : G \to \mathrm{PGL}(V_D^\vee).$$

The right side is an *affine* algebraic group. By choosing (non-canonically) a realization of this as a closed algebraic subgroup of some $\mathrm{GL}(W_D)$, we'll have the representations we've been looking for.

In order to define an action of $G(k)$ on $\mathbf{P}(V_D^\vee)(k)$, we consider the automorphism ℓ_g^* on $k(G)$ induced by 'composition with' $\ell_g : G \simeq G$ (for $g \in G(k)$). Identifying $k(U) = k(G)$, we rename this automorphism as L_g^* on $k(U)$. We want to compute $L_g^*(f_{g'}) \in k(U)^\times$ for $g, g' \in G(k)$. This rational function will be studied by looking at its associated Weil divisor. Note that $\mathrm{div}(f_{g'}) = (D_2)_{g'} - D_1$ on U has all of its generic points inside of the open $G \subseteq U$, and more precisely is equal to the extension by zero of $\ell_{g'}^{-1}(D) - D$. Thus, it is clear that $\mathrm{div}(L_g^*(f_{g'}))$ is the extension by zero of

$$\mathrm{div}(\ell_g^*(f_{g'}|G)) = \ell_g^{-1}(\mathrm{div}(f_{g'})|G) = \ell_g^{-1}(\ell_{g'}^{-1}(D) - D) = (\ell_{g'g}^{-1}(D) - D) - (\ell_g^{-1}(D) - D).$$

The extension by zero of this from G to U is exactly $((D_2)_{g'g} - D_1) - ((D_2)_g - D_1)$, so we have an equality of Weil divisors $\mathrm{div}(L_g^*(f_{g'})) = \mathrm{div}(f_{g'g}) - \mathrm{div}(f_g)$ on U. We conclude that there exists a constant $c(g, g') \in k^\times$ (depending heavily on our choices of f_γ for $\gamma \in G(k)$) such that

$$f_{gg'} = c(g, g') L_g^*(f_{g'}) f_g$$

inside of $k(U)^\times$.

The conclusion we can draw from this is that the intrinsic finite-dimensional space V_D is *stable* under the *injective* action $h \mapsto L_g^*(h) f_g$ on $k(U)$. Thus, for each $g \in G(k)$ and choice of f_g, we get an injective k-linear map $V_D \to V_D$ which therefore must be an automorphism (since $\dim_k V_D$ is finite). We label the induced automorphism of $\mathbf{P}(V_D^\vee)(k)$ by $\rho_D(g^{-1})$. Clearly $\rho_D(g^{-1})$ is unaffected by $k^\times$-scaling on f_g. Since the f_g's span V_D over k, it is readily checked that $\rho_D(g_1 g_2) = \rho_D(g_1)\rho_D(g_2)$ (this is why we put in that extra inverse, and by passing to projective spaces we remove the influence of the non-canonical $c(g, g')$'s and f_g's).

Lemma 4.1. *The map $G(k) \to \mathbf{P}(V_D^\vee)(k)$ sending g to $[f_g]$ is induced by an algebraic map $b_D : G \to \mathbf{P}(V_D^\vee)$.*

This algebraicity lemma will be proven in the next section. Let us now explain how to use it to prove that the ρ_D are induced by algebraic maps $G \to \mathrm{PGL}(V_D^\vee)$ (necessarily algebraic group maps). We'll then study the kernels as D varies.

It is easy to compute that $\rho_D(g)([f_{g'}]) = [f_{g'g^{-1}}] = b_D(g'g^{-1})$. Thus, if we fix a basis $f_{g_0}, \dots, f_{g_n}$ of V_D, then the algebraicity of the group structure on G ensures that the maps $G(k) \mapsto \mathbf{P}(V_D^\vee)(k)$ given by $g \mapsto \rho_D(g)([f_{g_i}])$ are algebraic maps. We therefore get an algebraic $(n+1)$-fold product map

$$G \to \mathbf{P}(V_D^\vee) \times_k \cdots \times_k \mathbf{P}(V_D^\vee)$$

which factors through a map $\psi_D : G \to B$ to the open subset B whose k-rational points correspond to $(n+1)$-tuples in $\mathbf{P}(V_D^\vee)(k)$ giving a 'projective basis'. We can therefore define an algebraic 'change of basis' map $B \to \mathrm{PGL}(V_D^\vee)$ (depending on the f_{g_i}'s in V_D) whose composite with ψ_D yields an algebraic map $G \to \mathrm{PGL}(V_D^\vee)$ which on k-rational points is ρ_D. This proves the algebraicity of ρ_D (granting Lemma 4.1, to be proven later).

What is the kernel K_D of ρ_D? Well, $K_D(k)$ consists of those $g \in G(k)$ for which $\mathrm{div}(f_{g'g}) = \mathrm{div}(f_{g'})$ on U for all $g' \in G(k)$. Taking $g' = e$ and intersecting our Weil divisors on U with the open subset G, we see that any $g \in K_D(k)$ satisfies $\ell_g^{-1}(D) = D$. We claim that for any $g \in G(k)$, $g \neq e$, there exists a codimension 1 closed subvariety D in G with $\ell_g^{-1}(D) \neq D$. To see this, choose any nonempty open affine T in G. Since G is irreducible, $Tg^{-1} \cap T$ is non-empty. Choose any k-rational point g_0 in here, so e and g lie in the open affine variety $g_0^{-1}T$ as distinct points. Thus, there is a codimension 1 closed subvariety in $g_0^{-1}T$ which passes through g and not e. Taking the closure of this in G, we get a codimension 1 closed subvariety D in G for which $\ell_g^{-1}(D) \neq D$. It follows that set-theoretically $\bigcap K_D = \{e\}$, where the intersection is taken over all effective Weil divisors D on G.

Since the space of G is noetherian, there exist finitely many such divisors $D_1, \dots, D_r$ on G such that $K_{D_1} \cap \dots \cap K_{D_r} = \{e\}$ set-theoretically. Thus, the algebraic group map

$$\rho = \rho_{D_1} \times \dots \times \rho_{D_r} : G \to \mathrm{PGL}(V_{D_1}^\vee) \times_k \dots \times_k \mathrm{PGL}(V_{D_r}^\vee)$$

has scheme-theoretic kernel finite over k (since it is supported at a single point). Thus, by first factoring ρ through the closed algebraic subgroup $\rho(G)$ we deduce via Lemma 2.1 that ρ must be finite (hence affine). Since the target of ρ is an affine algebraic group over k, we conclude that G is affine over k. This completes the proof of Chevalley's Theorem, up to proving the assertion in Lemma 4.1 that the set-theoretic map $G(k) \to \mathbf{P}(V_D^\vee)(k)$ given by $g \mapsto [f_g]$ is algebraic.

5. An Algebraicity Check

In this final section, we prove that $g \mapsto [f_g] \in \mathbf{P}(V_D^\vee)(k)$ is algebraic. This will require some care to do rigorously, though it ultimately comes down to correctly interpreting the universal mapping property of projective space in our particular setting, as well as making judicious use of the See-saw Theorem [9, Cor 6, p. 54].

Consider the inclusion of invertible sheaves on $G \times U$

$$\mathcal{L}(G \times D_1 - D_2) \hookrightarrow \mathcal{L}(G \times D_1) = p_2^*(\mathcal{L}(D_1)).$$

Roughly speaking, the left side consists of the family $\{\mathrm{div}(f_g)\}_{g \in G(k)}$ and the right side is the family $\{D_1\}_{g \in G(k)}$. To simplify the notation a little, let $\delta = G \times D_1 - D_2$, a Weil divisor on $G \times U$. Let $\overline{\mathcal{L}}$ be the *coherent* pushfoward of $\mathcal{L}(\delta)$ from $G \times U$ to $G \times X$ (note that the complementary codimension here is still at least 2, and the variety $G \times X$ is *normal* either by using EGA IV$_2$, 6.14.3, or the fact that $G \times X \to X$ is smooth and X is normal). We claim that there exists a non-empty open W in G such that $\overline{\mathcal{L}}|(W \times X)$ is invertible. This claim is reasonable because on each fiber over $g \in G(k)$ the sheaf $\mathcal{L}(\delta)$ induces the trivial sheaf on U, which has trivial (hence invertible) pushfoward to the normal X (imagine that pushfoward commutes with passage to fibers, which is roughly what we'll prove in our specific situation).

In order to construct a proof of invertibility of the pushfoward $\overline{\mathcal{L}}$ on some $W \times X$, we prove something a little more general.

Theorem 5.1. *Let Y and X be normal varieties over k, $U \subseteq X$ a non-empty open subvariety with complementary codimension at least 2. Let $\mathcal{L}$ be an invertible sheaf on $Y \times U$, $i : Y \times U \hookrightarrow Y \times X$ the inclusion, $i_y : U_y \hookrightarrow X_y$ the pullback by $\mathrm{Spec}(k(y)) \to Y$ for $y \in Y$. Assume that that induced invertible sheaf $\mathcal{L}_y$ on U_y is trivial for all $y \in Y(k)$. Then there exists a non-empty open set $W \subseteq Y$ such that $\mathcal{L}|W \times U$ extends to an invertible sheaf on $W \times X$.*

The proof Theorem 5.1 is somewhat of a digression, so we postpone it until the end.

Using Theorem 5.1, $\overline{\mathcal{L}}|W \times X$ is an invertible sheaf on $W \times X$ and has trivial fiber over each $g \in W(k)$, for some open non-empty $W \subseteq G$ (recall that an *invertible* sheaf on $X = (W \times X)_g$ with trivial restriction to U must itself be trivial). Since X is a *complete* variety, we can apply the See-saw Theorem to conclude that for the projection $q_1 : W \times X \to W$, $\mathcal{N} \stackrel{\text{def}}{=} (q_1)_*(\overline{\mathcal{L}}|W \times X) \simeq (p_1|W \times U)_*(\mathcal{L}(\delta)|W \times U)$ is an invertible sheaf on W and the natural map

$$q_1^*(\mathcal{N}) = q_1^*((q_1)_*(\overline{\mathcal{L}}|W \times X)) \to \overline{\mathcal{L}}|W \times X$$

is an isomorphism. Restricting this to $W \times U \subseteq W \times X$, the natural map $(p_1|W \times U)^*(\mathcal{N}) \to \mathcal{L}(\delta)|W \times U$ is an isomorphism.

Now we come back to the inclusion $\mathcal{L}(\delta) \hookrightarrow \mathcal{L}(G \times D_1) \simeq p_2^*(\mathcal{L}(D_1))$ on $G \times U$. By the Künneth formula with $V = \Gamma(U, \mathcal{L}(D_1))$ (a finite-dimensional k-vector space which contains V_D), the canonical map on G

$$\mathcal{O}_G \otimes_k V \to (p_1)_*(\mathcal{L}(G \times D_1)) = (p_1)_*(p_2^*(\mathcal{L}(D_1)))$$

is an isomorphism. Thus, applying $(p_1)_*$ to $\mathcal{L}(\delta) \hookrightarrow p_2^*(\mathcal{L}(D_1))$ and restricting to W yields an inclusion

$$j : \mathcal{N} \hookrightarrow \mathcal{O}_W \otimes_k V$$

of locally free $\mathcal{O}_W$-modules, compatible with the map from each side to the constant sheaf attached to $k(W \times U)$ on W. Moreover, applying $(p_1|W \times U)^*$ to j yields a composite map

$$\mathcal{L}(\delta)|W \times U \simeq (p_1|W \times U)^*(\mathcal{N}) \stackrel{(p_1|W \times U)^*(j)}{\hookrightarrow} \mathcal{O}_{W \times U} \otimes_k V \to \mathcal{L}(G \times D_1)|W \times U$$

which is our original inclusion (restricted to $W \times U$).

We claim that j is locally split, which is to say that for each $g \in W(k)$, $j \bmod \mathfrak{m}_g$ is non-zero. Base changing from $\mathrm{Spec}(k(g))$ to $\{g\} \times U$, we want to check that $\mathcal{L}(\delta)_g \to \mathcal{O}_U \otimes_k V$ is non-zero. Composing with $\mathcal{O}_U \otimes_k V \to \mathcal{L}(D_1)$, it suffices to check that the inclusion

$$\mathcal{L}(G \times D_1 - D_2) \hookrightarrow \mathcal{L}(G \times D_1)$$

over $G \times U$ induces a non-zero map on $\{g\} \times U$. Equivalently, we can study the inclusion $\mathcal{I}(D_2) = \mathcal{L}(-D_2) \to \mathcal{O}_{G \times U}$. But the fact that D_2 does not contain $\{g\} \times U$ (or, what is better, that $D_2 \hookrightarrow G \times U$ is an effective Cartier divisor relative to G) ensures the map $\mathcal{I}(D_2)_g \to \mathcal{O}_{\{g\} \times U}$ is non-zero, and even injective. This proves that j is locally split.

Thus, we can dualize to get a *surjection* $\mathcal{O}_W \otimes_k V^\vee \twoheadrightarrow \mathcal{N}^\vee$, with the target an invertible sheaf on W. This gives rise to an algebraic map $W \to \mathbf{P}(V^\vee)$. On k-rational points, we claim that $g \in W(k)$ goes over to $[f_g] \in \mathbf{P}(V^\vee)(k)$, which is (up to $k^\times$-scaling) the non-zero linear map $V^\vee \to k$ given by evaluation on the non-zero $f_g \in V_D \subseteq V$. Since the image of the map $\mathcal{L}(\delta)_g \simeq p_1^*(\mathcal{N}/\mathfrak{m}_g) \to \mathcal{O}_U \otimes_k V$ modulo any $\mathfrak{m}_u$ (for $u \in U(k)$) is the k-span of f_g in V, the determination of the image of $g \in W(k)$ in $\mathbf{P}(V^\vee)(k)$ follows readily from the definitions.

We conclude that our algebraic map $W \to \mathbf{P}(V^\vee)$ factors through the closed subvariety $\mathbf{P}(V_D^\vee)$ (since this can be checked on the level of k-rational points), so we have an algebraic map $b_D : W \to \mathbf{P}(V_D^\vee)$ which is exactly what we expect on the level of k-rational points. A glance back at the previous section shows that this is enough to conclude that ρ_D is algebraic on a non-empty open set (namely, on the intersection of the $g_i W^{-1}$). But ρ_D is a group map between algebraic groups, so it must therefore be algebraic everywhere, which is what we really wanted. Now that ρ_D is known to be everywhere algebraic, we even conclude that $g \mapsto [f_g] = \rho_D(g^{-1})([f_e])$ is 'everywhere algebraic'.

It remains to prove Theorem 5.1.

Proof. Since $Y \times X$ is normal (by EGA IV_2, 6.14.3) and $Y \times U$ has complementary codimension at least 2 in $Y \times X$, $i_*(\mathcal{L})$ is at least coherent and our assertion is equivalent to the invertibility of $i_*(\mathcal{L})|W \times X$.

Let $\xi \in Y$ be the generic point. If $(i_*(\mathcal{L}))_\xi$ on X_ξ is invertible, then by EGA IV_3, 8.5.5 and the *coherence* of $i_*(\mathcal{L})$, it follows that $i_*(\mathcal{L})|W \times X$ is invertible for some open W around ξ. By EGA IV_3, 9.4.7, it is enough to show that $(i_*(\mathcal{L}))_y$ is invertible on $X_y = X$ for all $y \in W(k)$, for some non-empty open $W \subseteq Y$. Consider the natural base change map

$$B_y : (i_*(\mathcal{L}))_y \to (i_y)_*(\mathcal{L}_y)$$

for $y \in Y$. Since X is normal and U has complementary codimension at least 2 in X and $\mathcal{L}_y$ is trivial for all $y \in Y(k)$ by hypothesis (!), it follows that the right side of the base change map is trivial whenever $y \in Y(k)$. Thus, all we need to do is find a non-empty open W in Y so that $i_*(\mathcal{L})|W \times X$ is of formation compatible with arbitrary base change over W.

More generally, if $Z \to Y$ is a finite type map of noetherian schemes with Y integral and $i : U \hookrightarrow Z$ is an open subscheme and $\mathcal{F}$ is a coherent $\mathcal{O}_U$-module, then we will show that there exists a non-empty open W in Y so that the (quasi-coherent) $\mathcal{O}_Z$-module $i_*(\mathcal{F})$, restricted to the part of Z lying over W, is of formation compatible with arbitrary base change *over* W. We do not care about whether or not this $\mathcal{O}_Z$-module is coherent. The claim is clearly local on both Y and Z, so without loss of generality $Y = \mathrm{Spec}(B)$ is affine and $Z = \mathrm{Spec}(A)$ is affine. Cover U by finitely many basic opens $U_j = \mathrm{Spec}(A_{a_j})$, with $\mathcal{F}|U_j \simeq \widetilde{M_j}$ for a *finite* A_{a_j}-module M_j.

Letting $U_{j,j'} = U_j \cap U_{j'}$ and $i_{j,j'} : U_{j,j'} \hookrightarrow Y$ be the inclusion, we have an exact sequence

$$0 \to i_*(\mathcal{F}) \to \oplus (i_j)_*(\mathcal{F}|U_j) \xrightarrow{\varphi} \oplus (i_{j,j'})_*(\mathcal{F}|U_{j,j'}).$$

There are natural base change maps (over Y) defined for all of these pushfowards, and the formation of this complex (but possibly not its exactness!) respects the base change maps. Since i_j, $i_{j,j'}$, and $X \to Y$ are affine, the base change maps on the middle and right terms are isomorphisms. In order to handle the quasi-coherent kernel, it would be enough to prove that the (quasi-coherent) image of φ and the (quasi-coherent) cokernel of φ are Y-flat (which forces the exactness of the complex to be preserved by any base change). We will instead prove that this image is W-flat over a non-empty open W in Y, from which we get that the part of $i_*(\mathcal{F})$ lying over W is of formation compatible with any base change on W, which is what we wanted to prove.

The image of φ is the quasi-coherent sheaf attached to a *finite* module over $\prod A_{a_j}$, which is a finite type A-algebra, hence is a finite type B-algebra. By the Theorem on Generic Flatness [EGA IV$_2$, 6.9.1], there is a non-zero $b \in B$ so that over the non-empty open $W_1 = \mathrm{Spec}(B_b) \subseteq Y$ the image of φ is W_1-flat. A similar argument applies to the cokernel of φ (using $\prod A_{a_j a_{j'}}$), giving a suitable non-empty open $W_2 \subseteq Y$. Take $W = W_1 \cap W_2$. This finishes the proof of Theorem 5.1, Lemma 4.1, and Chevalley's Theorem on algebraic groups. ■

References

[1] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Springer-Verlag, 1990.
[2] C. Chevalley, *Une démonstration d'un théorème sur les groupes algébriques*, J. Mathématiques Pures et Appliqées, **39** (4), 1960, pp. 307–317.
[3] B. Conrad, *Deligne's notes on Nagata compactifications*, `www-math.mit.edu/~dejong`.
[4] A. Grothendieck, *Séminaire de géométrie algébrique* 3, Springer Lecture Notes 151.
[5] J. Humphreys, *Linear algebraic groups*, GTM 21, Springer-Verlag, 1975.
[6] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Mathematica **80** (1), 1993, pp. 95–111.
[7] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, 1986.
[8] J. Milne, *Jacobian varieties* in Arithmetic Geometry, Cornell/Silverman, ed., Springer-Verlag, 1986.
[9] D. Mumford, *Abelian varieties*, Oxford University Press, 1970.
[10] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto **2** (1), 1962, pp. 1–10.
[11] M. Nagata, *A generalization of the imbedding problem*, J. Math. Kyoto **3** (1), 1963, pp. 89-102.
[12] M. Rosenlicht, *Some basic theorems on algebraic groups*, American Journal of Mathematics **78**, 1956, pp. 401-443.
[13] J-P. Serre, *Algebraic groups and class fields*, GTM 117, Springer-Verlag, 1988.
[14] J-P. Serre, J. Tate, *Good reduction of abelian varieties*, Annals of Mathematics **88**, 1968, pp. 492–517.
[15] W. Waterhouse, *Introduction to affine group schemes*, GTM 66, Springer-Verlag, 1979.
[16] A. Weil, *Foundations of algebraic geometry*, American Math. Society, 1963.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: `bdconrad@umich.edu`